

Total quantum coherence and its applications

Chang-shui Yu · Si-ren Yang · Bao-qing Guo

Received: date / Accepted: date

Abstract Quantum coherence is the most fundamental feature of quantum mechanics. The usual understanding of it depends on the choice of the basis, that is, the coherence of the same quantum state is different within different reference framework. To reveal all the potential coherence, we present the total quantum coherence measures in terms of two different methods. One is optimizing maximal basis-dependent coherence with all potential bases considered and the other is quantifying the distance between the state and the incoherent state set. Interestingly, the coherence measures based on relative entropy and l_2 norm have the same form in the two different methods. In particular, we show that the measures based on the non-contractive l_2 norm is also a good measure different from the basis-dependent coherence. In addition, we show that all the measures are analytically calculable and have all the good properties. The experimental schemes for the detection of these coherence measures are also proposed by multiple copies of quantum states instead of reconstructing the full density matrix. By studying one type of quantum probing schemes, we find that both the normalized trace in the scheme of deterministic quantum

This work was supported by the National Natural Science Foundation of China, under Grant No.11375036, the Xinghai Scholar Cultivation Plan and the Fundamental Research Funds for the Central Universities under Grant No. DUT15LK35 and No. DUT15TD47.

Chang-shui Yu
School of Physics and Optoelectronic Technology, Dalian University of Technology, Dalian 116024, P. R. China
E-mail: quaninformation@sina.com; or ycs@dlut.edu.cn

Si-ren Yang
School of Physics and Optoelectronic Technology, Dalian University of Technology, Dalian 116024, P. R. China

Bao-qing Guo
School of Physics and Optoelectronic Technology, Dalian University of Technology, Dalian 116024, P. R. China

computation with one qubit (DQC1) and the overlap of two states in quantum overlap measurement schemes (QOM) can be well described by the change of total coherence of the probing qubit. Hence the nontrivial probing always leads to the change of the total coherence.

Keywords Quantum coherence · quantum entanglement · purity

PACS 03.67.Mn · 03.65.Ud

1 Introduction

Coherence is not only the essence of the interference phenomena but also the foundation of quantum theory [1]. It is almost directly or indirectly related to all the intriguing quantum phenomena. The most remarkable phenomena are quantum correlation including quantum entanglement. Both can be understood as the combination of the coherence and the tensor product structure of state space and play important roles in quantum information processing tasks (QIPs) [2-5]. In addition, it is also shown that quantum coherence has been widely applied in quantum thermal engine [6,7], biological system [8] and quantum parallelism [9].

In quantum information, quantum feature such as entanglement [10] and quantum correlation [11], due to the potential application in QIPs, can be well quantified from the resource theory point of view. Recently, In the same manner a rigorous framework has been developed for the quantification of quantum coherence [12]. It points out that a good coherence measure should satisfy three conditions: 1) The incoherent states have no coherence; 2) (Monotonicity) Incoherent completely positive and trace preserving maps cannot increase the coherence, or the average coherence is not increased under selective measurements; 3) (Convexity) It is not increased under the mixing of quantum states. Meanwhile it also presented several good coherence measures. However, such coherence measures strongly depend on the choice of the basis. This means that a quantum state can have certain coherence in one basis, but it could possess more, less, or none coherence in the other basis. Even though such a basis-dependent quantification of quantum coherence is consistent with our intuitive understanding (such as the contribution of off diagonal entries of density matrix), this could only consider the partial contribution of coherence of a state, once one is allowed to freely select the basis. In particular, the change of basis is a quite easy thing or at a small price at practical scenarios. Taking the linear optics for an example, one can only rotate the wave plate to get to another framework [13]. Since quantum coherence can be understood as the useful resource, why not try one's best to extract it as many as possible? So it is natural to consider, with all potential basis taken into account, how much coherence a state possesses or what is the maximal coherence in a state.

In this paper, we present the total coherence measure to quantify all the contributions of the quantum coherence in a state. The most distinct feature

of this measure is that it only covers the property of a state instead of the external observable (the choice of basis). We give several analytically calculable coherence measures in two different frameworks: optimization among all potential bases or quantifying the distance between the state and the incoherent state set. We find that all measures satisfy the mentioned three properties. In particular, one can find that the measure based on l_2 norm is also a valid candidate, even though the l_2 norm is not contractive. In addition, we find that the coherence measures based on relative entropy and the l_2 norm have the same result in the different frameworks. From the angle of the experimental detection, we give an explicit scheme to physically detect these measures. It is shown that such detections do not require reconstructing the full density matrix. As an application, we study the total coherence in the DQC1-like quantum probing schemes [14,15] including the QOM [16]. As we know, DQC1-like quantum schemes show quantum speedup, but what the source of the speedup is remains open. Here instead of finding the exact source, we study what cost is needed to pay for such schemes. It is found that both the normalized trace in DQC1 and the overlap of two states in QOM can be well described by the change of the total coherence of the probing qubit. In other words, the nontrivial quantum probing always gives rise to the change of the total coherence. The paper is organized as follows. We first propose various total coherence measures; then we present the properties of these measures; and then we study the total coherence in the DQC1-like quantum probing schemes; Finally, we give a summary and discussions.

2 The total coherence measure

The classical coherence is usually characterized by the frequencies and the phases of different waves, but a good definition of quantum coherence stemming from the superposition of state (a single wave) depends not only on the state itself but also on the associated observable. The physical root of such a definition is that the measurement on the observable can reveal the interference pattern provided that the observable does not commute with the considered density matrix. In this sense, it is obvious that the coherence measure will have to depend on the framework (or basis) that the density matrix is given in [17,18]. Therefore, there are naturally two ways to quantifying quantum coherence: one is based on the commutation, the other is based on the distance. Based on the former, Ref. [19] used the skew information to measure the coherence, and based on the latter, Ref. [12] proposed several measures. We also used l_1 norm to study the source of quantum entanglement [20]. Considering the potential classification of coherence of composite quantum system, we have provided a new angle to understand the geometric quantum discord, quantum non-locality and the monogamy of coherence [21]. Here we shall consider the maximal coherence with different bases taken into account, or the total coherence which a state could have. Therefore, a natural method to doing so is to maximize the basis-dependent coherence by taking into account all the

potential bases. In addition, as mentioned before, the coherence can be embodied by the commutation between the state and some particular observable. Since we consider all potential bases or (observables), it is implied that the incoherent state requires that the density matrix should commute with all observables. The direct conclusion is that the incoherent state is the maximally mixed state $\frac{1}{n}$ with n denoting the dimension of the state and $\mathbf{1}_n$ denoting the n -dimensional unity. So one can easily construct the coherence measure based on the 'distance' [9]. In the following, we will consider the coherence measures both by optimizing the basis and by the distance.

Coherence based on basis optimization.-With the different bases considered, we can define the total coherence based on the optimization of basis as follows.

$$C(\rho) = \max_U \|U\rho U^\dagger - \sigma_U\|, \quad (1)$$

where $(\sigma_U)_{ii} = (U\rho U^\dagger)_{ii}$ denotes the diagonal matrix and $\|\cdot\|$ denotes some good norms or distance functions. For example, we can employ the l_1 norm, l_2 norm, relative entropy and so on. One can also use the trace norm and Fidelity, but the incoherent state σ_U is usually not given by $(U\rho U^\dagger)_{ii}$, but some particular states in the incoherent set [12]. The skew information can also be employed, but no explicit incoherent state is required. In order to provide an explicit expression of the total coherence, next we will list some coherence measures by the particular choice of the "norms".

(1) l_2 norm could be the most easily calculable norm. But it is not contractive, so in many cases an unphysical result could appear [12]. In the current case, one will find in the paper that l_2 norm can be safely used to quantify the total coherence measure. Based on l_2 norm, we have

$$\begin{aligned} C_2(\rho) &= \max_U \|U\rho U^\dagger - \sigma_U\|_2 \\ &= \text{Tr}\rho^2 - \min_{|i\rangle} \sum_{i=1}^n |\langle i|\rho|i\rangle|^2 \\ &= \text{Tr}\rho^2 - \frac{1}{n}. \end{aligned} \quad (2)$$

The minimum can be reached because there always exists the basis $\{|i\rangle\}$ such that the diagonal entries of ρ are uniform.

(2) If we employ the relative entropy [9], the total coherence can be given by

$$\begin{aligned} C_{re}(\rho) &= \max_U S(U\rho U^\dagger || \sigma_U) \\ &= \text{Tr}\rho \log \rho - \min_U \sum_{i=1}^n \sigma_U \log \sigma_U \\ &= \log n - S(\rho), \end{aligned} \quad (3)$$

with $S(\rho||\sigma) = \text{Tr}\rho \log \rho - \text{Tr}\rho \log \sigma$ denoting the relative entropy of ρ and σ . The minimum is also achieved by the basis subject to the uniform distribution of the diagonal entries of ρ .

(3) Based on skew information, we will have a different definition. The skew information [22-24] for a density matrix ρ and an observable K is given by $I(\rho, K) = -\frac{1}{2}\text{Tr}[\sqrt{\rho}, K]^2$. Based on the skew information, the total coherence can be defined by

$$\begin{aligned} C_I &= \max_{\{|k\rangle\}} \sum_{k=1}^n \left(\langle k | \rho | k \rangle - \langle k | \sqrt{\rho} | k \rangle^2 \right) \\ &= 1 - \min_{\{|k\rangle\}} \langle k | \sqrt{\rho} | k \rangle^2 \\ &= 1 - \frac{1}{n} \left(\sum \sqrt{\lambda_j} \right)^2, \end{aligned} \quad (4)$$

where λ_i is the eigenvalue of ρ . In particular, one can find that the minimum can always be reached when in the basis $\{|k\rangle\}$ the diagonal entries of $\sqrt{\rho}$ are uniform. Why C_I can quantify the coherence can be easily found as follows. Given a basis $\{|k\rangle\}$, ρ can be diagonalized by the basis $\{|k\rangle\}$, iff $[\sqrt{\rho}, |k\rangle\langle k|] = 0$ for all $|k\rangle$. So the skew information in the basis $\{|k\rangle\}$ can quantify the coherence of ρ in this basis. Considering all the potential basis, the maximal C_I naturally quantifies the total coherence as mentioned above. This definition should be distinguished from that in Ref. [19] where the coherence could depend on the eigenvalues of the observable.

l_1 norm of a matrix is defined by the sum of the absolute values of all the entries of the matrix. It is also a good norm even though it is not a unitary-invariant norm. With this norm, the total coherence can be given by $C_1(\rho) = \max_U \sum_{i \neq j} |\langle i | U \rho U^\dagger | j \rangle|$, with the maximum reached when all the diagonal entries of $U \rho U^\dagger$ equal $\frac{1}{n}$. However, because l_1 norm is not unitary-invariant, the optimal result of $C_1(\rho)$ for a general ρ (especially in high dimensional Hilbert space) cannot be easily given. But one can find that C_1 is unitary-invariant, because the optimization compensates for it. In addition, the explicit expressions of the total coherence based on trace norm and the Fidelity can not be easily given because the nearest incoherent state can not be determined in general cases.

Coherence based on distance.-Since the completely incoherent state is $\frac{\mathbf{1}_n}{n}$, one can always define the total coherence based on the distance between a given state and $\frac{\mathbf{1}_n}{n}$. Using some (unitary-invariant) norms or distance functions $\|\cdot\|$, we have

$$\tilde{C}(\rho) = \left\| \rho - \frac{\mathbf{1}_n}{n} \right\|. \quad (5)$$

Since no optimization is included, all the coherence measures can be easily calculated so long as one selects a proper function $\|\cdot\|$. For example, one can easily find the explicit form of the total coherence measure $\tilde{C}(\rho)$ based on trace norm, Fidelity and so on. Here, we would like to emphasize the following several candidates.

($\tilde{1}$) If the relative entropy is used, one can find

$$\begin{aligned}\tilde{C}_{re}(\rho) &= \text{Tr}\rho \log \rho - \text{Tr}\rho \log \frac{\mathbf{1}_n}{n} \\ &= \log n - S(\rho) = C_{re}(\rho).\end{aligned}\quad (6)$$

($\tilde{2}$) If l_2 norm is selected, we have

$$\begin{aligned}\tilde{C}_2(\rho) &= \left\| \rho - \frac{\mathbf{1}_n}{n} \right\|_2 \\ &= \text{Tr}\rho^2 - \frac{1}{n} = C_2(\rho).\end{aligned}\quad (7)$$

It is obvious that the total coherence measures based on the relative entropy and the l_2 norm have the same final expressions in the different frameworks. Because l_1 norm could be changed by a unitary operation, the coherence based on it has to take some optimization on the unitary transformations, that is, $\tilde{C}_1(\rho) = \max_U \left\| U\rho U^\dagger - \frac{\mathbf{1}_n}{n} \right\|_1$. Here we would like to emphasize that our coherence measures actually are closely related to the purity. One knows that the purity of a state ρ is defined by $P(\rho) = \text{Tr}\rho^2$. It is obvious that $P = 1$ for pure states and $\frac{1}{n} \leq P < 1$ for mixed state, based on which one can design many other similar quantities for purity such as $1 - S(\rho)$ and $(\sum_i \sqrt{\lambda_i})^n$ with $n \geq 1$ and so on. These purities reach maximum value for pure state and nonzero minimum value for maximally mixed states. Thus the presented total coherence can be regarded as a displacement on the purity. In this sense, we give the purity a new understanding by the coherence and *vice versa*.

One should note that our coherence measures are defined different from the basis-dependent coherence, so the criteria for a good measure should be different either. Next, we will list the useful properties that these measures satisfy, meanwhile they could form new criteria for a basis-independent coherence measure.

3 Properties

In what follows, we will list several good properties that our above total coherence measures satisfy. In particular, we will show that the coherence measure based on l_2 norm is still a monotone, even though it is not a contractive norm. This could provide great convenience for the future applications.

(I) *Maximal for pure states and vanishing for incoherent states.*- It is easy to find that all the coherence measures vanish for maximally mixed state $\frac{\mathbf{1}_n}{n}$ and arrive at its maximal value for pure states. This can be well understood, since any pure state can be converted to a maximally coherent state by changing basis.

(II) *Invariant under unitary operations.*-The most obvious feature, based on the definitions, is that all these measures are invariant under unitary transformations.

(III) *Convexity*.- All the measures are convex. That is, the total coherence will not increase under mixing. This can be found from the fact that all the norms satisfy the triangle inequality. For the squared l_2 norm, one also needs to consider the convexity of the quadratic function. In addition, we know that the fidelity is strongly concave, the von Neumann entropy is concave and the skew information is convex, so this property can be easily proved.

(IV) *Monotonicity*.- This property will have different contents from that for the basis-dependent coherence measure. Just as in entanglement theory and coherence measure [12], the definitions of entanglement monotone and coherence monotone require the non-entangling operations and the basis-dependent incoherent operations, respectively. Now let the incoherent operation be given in Kraus representation as $\$K = \{K_n | \sum K_n^\dagger K_n = \mathbf{1}\}$. If we follow the rules of non-entangling operations and the basis-dependent incoherent operations the elements of which cannot generate entanglement or coherence, one can easily find that K_n should be a unitary transformation neglecting a constant (corresponding to probability). Therefore, here the incoherent operations can be rewritten as $\$U = \{U_n | \sum p_n U_n^\dagger U_n = \mathbf{1}, U_n^\dagger U_n = \mathbf{1}\}$. A simple algebra can show that the average total coherence equals the total coherence of the original state before the operation, but the total coherence of the final state after the operation will not be increased due to the convexity. We would like to emphasize that the coherence measure based on l_2 norm also satisfies this property, so it can be used safely.

(V) *Coherence doesn't increase under the special POVM*.- This is another interesting property for the total coherence. Let's consider such an operation that is given in Kraus representation as $\$I = \{K_n | \sum K_n^\dagger K_n = \sum K_n K_n^\dagger = \mathbf{1}\}$. One can find that this operation can not create any coherence from the incoherent state $\frac{\mathbf{1}_n}{n}$, even though the single element such as K_n may produce coherence. Therefore, the average total coherence could be increased by $\$I$. However, it is interesting that the total coherence of the final state (the final ensemble generated by $\$I$ on the original state) after this operation can not be increased. This conclusion may be drawn from the fact that all the above employed quantifications but l_2 norm are contractive. However, one can also prove that this property is satisfied for the case of l_2 norm. The proof is given as follows.

Let ρ denote the density matrix that we want to consider. The final state after the operation can be given by $\tilde{\rho} = \sum_{ij} K_i \rho K_i^\dagger$. Thus $Tr \tilde{\rho}^2 = \sum_{ij} K_i \rho K_i^\dagger K_j \rho K_j^\dagger = Tr \sum_m \Lambda A_m \Lambda A_m^\dagger$, where we use the eigenvalue decomposition of $\rho = U \Lambda U^\dagger$ with Λ denoting the diagonal matrix of eigenvalues λ_i and $A_m = U^\dagger K_i^\dagger K_j U$ with $m = (ij)$. Since $\sum K_n^\dagger K_n = \sum K_n K_n^\dagger = \mathbf{1}$, it is obvious that $\{A_m\}$ defines a Positive Operator-Valued Measurement (POVM) and $\sum_m A_m^\dagger A_m = \sum_m A_m A_m^\dagger = \mathbf{1}$ which implies $\sum_{jm} |[A_m]_{ij}|^2 = \sum_{im} |[A_m]_{ij}|^2 = 1$. Expand

$Tr\tilde{\rho}^2$, we will have

$$\begin{aligned}
Tr\tilde{\rho}^2 &= \sum_{ijm} |[A_m]_{ij}|^2 \lambda_i \lambda_j \\
&\leq \left[\sum_i \lambda_i^2 \right]^{1/2} \left[\sum_i \left(\sum_{jm} |[A_m]_{ij}|^2 \lambda_j \right)^2 \right]^{1/2} \\
&\leq \left[\sum_i \lambda_i^2 \right]^{1/2} \left[\sum_{ij} \sum_m |[A_m]_{ij}|^2 \lambda_j^2 \right]^{1/2} \\
&= \sum_i \lambda_i^2 = Tr\rho^2.
\end{aligned} \tag{8}$$

This shows that the total coherence based on l_2 norm is not increased under the operation $\$_I$ either.

4 Measurable total coherence

In the above sections, we mainly consider the mathematical approaches to measuring the total coherence. How can we directly measure the coherence experimentally? In fact, one can note that the presented measures can be expressed by the function of the eigenvalues of the density matrix ρ , for example, Eqs. (2,3,4,6,7). Since the eigenvalues of a density can be directly measured (for example, the schemes for the measurable entanglement and discord [25-27]), our presented coherence can be naturally determined. However, for integrity and the latter use, we would like to briefly describe the concrete implementation. Since $Tr\rho^n = \sum_{i=1}^N \lambda_i^n$ for any N -dimensional density matrix ρ , one can only set $n = 1, 2, \dots, N$, respectively, and experimentally measure $Tr\rho^n$. In this way, we can get N equations depending on the N eigenvalues. In principle, all the eigenvalues can be determined by solving these equations. In order to do so, we can define the generalized swapping operator as $V_n |\psi_1, \psi_2, \dots, \psi_n\rangle = |\psi_n, \psi_1, \psi_2, \dots, \psi_{n-1}\rangle$. With the swapping operator one can find $Tr\rho^n = TrV_n \rho^{\otimes n}$. Thus one can first prepare a probing qubit $|\varphi\rangle_p = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and n copies of measured state ρ . Then let the $n+1$ particles undergo a controlled V_n gate, i.e., $\mathbf{1}_2 \oplus V_n$ with the probing qubit as the control qubit. Finally, the σ_x measurement is performed on the probing qubit and the probability of obtaining ± 1 will be $\frac{1 \pm Tr\rho^n}{2}$ which is as expected. The quantum circuit is shown in Fig. 1. Hence, generally speaking, all the coherence measures can always be obtained by measuring $N-1$ V_n with at most N copies of the state ρ . However, for the coherence measure based l_2 norm, one can find that the measurement scheme becomes quite simple, because it can be directly obtained by only measuring V_2 with only 2 copies of ρ , which does not depend on the dimension of the measured density matrix. This is akin to the overlap measurement scheme [16].

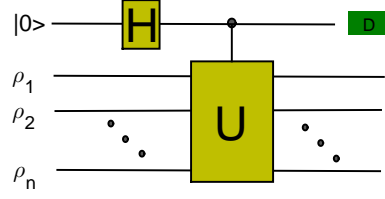


Fig. 1 (color online) The circuit for the scheme of measurable total coherence and the similar probing scheme. This is a generalized QOM with $U = V_n$. For the quantum probing scheme, the initial state of the probing qubit is given by ρ_p and the initial state $\rho_1 \otimes \rho_2 \cdots \rho_n$ is usually replaced by some density matrix ρ_s . In DQC1 scheme, $\rho_s = \frac{1}{N}$ with $N = 2^n$.

5 The cost of DQC1-like quantum probing schemes

Here we consider the DQC1-like quantum probing schemes which include the above mentioned QOM [16] and the remarkable DQC1 schemes [14,15]. The features of this kind of schemes are (i) a probing qubit is used to extract the information from the quantum system; (ii) the cost of the probing does not depend on the probed quantum systems (or the dimension of the input state space), once the system has been designed. The quantum circuit can be sketched as Fig. 1, where a probing qubit is sent to the probed quantum system, and the interaction between the probing qubit and the system is usually provided by one or several controlled- U operations. It is shown that there exists quantum speedup in the schemes [14,15], but the essence of this speedup is neither entanglement nor discord between the probing qubit and the probed qubits [14,28,29]. Most people could think that the coherence as the candidate should be an intuitive physics, but no quantitative description has been presented up to now. Here we will do such a job by proving a weak result that nontrivial probing needs the existence of coherence.

For generality, we set the probing qubit to be given by

$$\rho_p = \frac{1}{2} (\mathbf{1}_2 + \mathbf{P} \cdot \vec{\sigma}) \quad (9)$$

where \mathbf{P} is a real 3-dimensional vector with $|\mathbf{P}| \leq 1$ and $\vec{\sigma}$ denotes the vector made up of the 3 Pauli matrices. Suppose that the probed n -dimensional density matrix is denoted by ρ_s . So the controlled- U operation will lead to the final state as

$$\rho_f = (\mathbf{1}_n \oplus U) (H \rho_p H \otimes \rho_s) (\mathbf{1}_n \oplus U^\dagger), \quad (10)$$

with H the Hadamard gate. Thus the final density matrix of the probing qubit becomes

$$\rho_{pf} = \frac{1}{2} \begin{pmatrix} 1 + P_1 & (P_3 + iP_2) \text{Tr} \rho_s U^\dagger \\ (P_3 - iP_2) \text{Tr} \rho_s U & 1 - P_1 \end{pmatrix}. \quad (11)$$

In order to guarantee that this probing scheme works, it is required that P_2 and P_3 do not vanish simultaneously. Similarly, if we want to use U operation to probe information of ρ_s , U should not be the identity. Based on Eq. (2)

and Eq. (10), one can easily obtain the total coherence based on l_2 norm for ρ_p and ρ_{pf} as

$$C(\rho_p) = \frac{|\mathbf{P}|^2}{2}, \quad (12)$$

$$C(\rho_{pf}) = \frac{1}{2} \left[P_1^2 + (P_2^2 + P_3^2) |Tr \rho_s U|^2 \right]. \quad (13)$$

So the change of the total coherence can be given by

$$\Delta C_U = |C(\rho_{pf}) - C(\rho_p)| = \frac{(P_2^2 + P_3^2)}{2} (1 - |Tr \rho_s U|^2), \quad (14)$$

with the subscript U denoting the change of the total coherence induced by the controlled- U operation. Thus ΔC_U is closely related to the evaluation of this quantum probing scheme. From the point of probing qubit of view, the cost of the probing qubit is that the total coherence changes ΔC_U for such a task. Generally, if U is an identity which means we do nothing in the scheme, one will see that $\Delta C_U = 0$. For the general DQC1 where $P_1 = P_2 = 0$ and $\rho_s = \frac{1_N}{N}$ with $N = 2^n$, we have $\Delta C_U = \frac{P_3^2}{2} \left(1 - \left| \frac{Tr U}{2^n} \right|^2 \right)$ which is consistent with Ref. [30]. For the QOM where $P_1 = P_2 = 0$, $P_3 = 1$ and $\rho_s = \rho_1 \otimes \rho_2$, one can immediately find that $\Delta C_{V_2} = \frac{1}{2} (1 - |Tr \rho_1 \rho_2|^2)$ which is directly given by the overlap of ρ_1 and ρ_2 .

In fact, the above probing schemes maybe include more unitary operations denoted by U_i (Here we mainly consider the controlled- U operations, and the unitary operations separately performed on the probing qubit and the probed quantum state. In particular, it is more reasonable to consider all the operations given by the basic quantum logic gates.). We would like to define the cost of such a scheme \mathcal{C} as the sum of the changes of the total coherence of the probing qubit. That is,

$$\mathcal{C} = \sum_k \Delta C_k \quad (15)$$

with k taking all the unitary operations. In particular, one should note that $\Delta C_k \geq 0$ for any k based on Eq. (15). Therefore, generally speaking, the more operations are used, the more cost is paid. In particular, \mathcal{C} will not vanish if the probing scheme only includes two unitary operations such as U and U^\dagger . This should be distinguished from the scheme which only includes a single identity operation. This difference can be understood in the frame of basic logic gates. That is, suppose U and U^\dagger are given by a series of logic gates, the qubits through them will have to undergo the corresponding ‘dynamical evolution’, even though at the final moment the original state is recovered. On the contrary, a direct identity operation (we mean no logic gates), no such an ‘evolution’ is needed. In this sense, the subscript k in Eq. (15) taking all the covered logic gates could be more reasonable, but it will lead to more complicated calculations because how to construct a given operation by logic gates has to be considered.

Finally, one can find that the coherence measures given by Eqs. (3,4) are described by the eigenvalues of the density matrix. In the above probing scheme, one can easily calculate that the eigenvalues of the density matrix ρ_{pf} are the functions of $(P_2^2 + P_3^2) |Tr\rho_s U|^2$. Hence we can always find what the cost \mathcal{C} is for the different measures which we can choose. In particular, it can be shown that \mathcal{C} is directly related to $(P_2^2 + P_3^2) |Tr\rho_s U|^2$. To sum up, \mathcal{C} vanishes if and only if the probing scheme is trivial. In this sense, we think that \mathcal{C} can be understood as the cost of such a probing scheme.

6 Discussion and conclusion

We have studied the total coherence of a quantum state and presented several coherence measures which are independent of the basis. It is shown that all the presented measures especially including the measure based on l_2 norm satisfy all properties such as the monotonicity. This actually provides a very convenient tool for the relevant researches due to the simple form of l_2 norm. In particular, we have shown that the total coherence measures based on the relative entropy and the l_2 norm have the same expression by optimizing the basis or by quantifying the distance. In addition, for integrity, the experimental schemes for the detection of coherence are also briefly introduced. Finally, we study the total coherence in the DQC1-like quantum probing schemes. It is shown that both the normalized trace in DQC1 and the overlap of the two states in QOM can be well described by the change of the total coherence of the probing qubit. In other words, all the nontrivial probing schemes have to lead to the change of the total coherence. Therefore, this change can be understood as the cost of implementing such a probing scheme. This could motivate a new platform to study the essence of the speedup of mixed-state quantum computing.

References

1. Winter, R. G., Steinberg, A. M., Attwood, D.: *Coherence. Accessscience* (McGraw-Hill, 2008). Available at: <http://accessscience.com/content/coherence/146900>.
2. Horodecki, R. *et al.*: Quantum entanglement. *Rev. Mod. Phys.* **81**, 865 (2009).
3. Henderson, L. , Vedral V.: Classical, quantum and total correlations. *J. Phys. A* **34**, 6899 (2001).
4. Vedral, V.: Classical correlations and entanglement in quantum measurements. *Phys. Rev. Lett.* **90**, 050401 (2003).
5. Ollivier, H. , Zurek, W. H.: Quantum discord: a measure of the quantumness of correlations. *Phys. Rev. Lett.* **88**, 017901 (2001).
6. Scully, M. O. *et al.*: Extracting work from a single heat bath via vanishing quantum coherence, *Science* **299**, 862 (2003).
7. Scully, M. O. *et al.*: Quantum heat engine power can be increased by noise-induced coherence. *Proc. Natl. Acad. Sci. U. S. A.* **108**, 15097 (2011).
8. Huelga, S. F. , Plenio, M. B.: Vibrations, quanta and biology. *Contemp. Phys.* **54**, 181 (2013).
9. Nielsen, M. A. , Chuang, I. L.: *Quantum Computation and Quantum Information* Ch.1, 30-38 (Cambridge University Press, Cambridge, 2000).

10. Plenio, M. B. , Virmani, S.: An Introduction to entanglement measures. *Quant. Inf. Comp.* **7**, 1 (2007).
11. Modi, K. *et al.*: The classical-quantum boundary for correlations: Discord and related measures. *Rev. Mod. Phys.* **84**, 1655 (2012).
12. Baumgratz, T., Cramer, M. , Plenio, M. B.: Quantifying coherence. *Phys. Rev. Lett.* **113**, 140401 (2014).
13. Walls, D. F. , Milburn, G. J.: *Quantum Optics* Ch.16, 297-303 (Springer- Verlag, Berlin Heidelberg, 1994).
14. Datta, A., Shaji, A. , Caves, C. M.: Quantum discord and the power of one qubit. *Phys. Rev. Lett.* **100**, 050502 (2008).
15. Knill, E. , Laflamme, R.: Power of one bit of quantum information. *Phys. Rev. Lett.* **81**, 5672 (1998).
16. Filip, R.: Overlap and entanglement-witness measurements. *Phys. Rev. A* **65**, 062320 (2002).
17. Cunha, M. O. T.: The geometry of entanglement sudden death. *New J. Physics* **9**, 237 (2007).
18. Ann, K. , Jaeger, G.: Finite-time destruction of entanglement and non-locality by environmental influences. *Foundations of Physics* **39**, 790 (2009).
19. Girolami, D.: Observable measure of quantum coherence in finite dimensional systems. *Phys. Rev. Lett.* **113**, 170401 (2014).
20. Yu, C. S. , Song, H. S.: Bipartite concurrence and localized coherence. *Phys. Rev. A* **80**, 022324 (2009).
21. Yu, C. S., Zhang, Y. , Zhao, H. Q.: Quantum correlation via quantum coherence. *Quant. Inf. Proc.* **13**, 1437 (2014).
22. Wigner, E. P. , Yanase, M. M.: Information content of distribution. *Proc. Natl. Acad. Sci. U.S.A.* **49**, 910 (1963).
23. Luo, S.: Wigner-Yanase Skew information and uncertainty relations. *Phys. Rev. Lett.* **91**, 180403 (2003).
24. Girolami, D., Tufarelli, T. , Adesso, G.: Characterizing nonclassical correlations via local quantum uncertainty. *Phys. Rev. Lett.* **110**, 240402 (2013).
25. Horodecki, P. , Ekert, A.: Method for direct detection of quantum entanglement. *Phys. Rev. Lett.* **89**, 127902 (2002).
26. Cai, J. M. , Song, W.: Novel schemes for directly measuring entanglement of general states. *Phys. Rev. Lett.* **101**, 190503 (2008).
27. Jin, J. S., *et al.*: Direct scheme for measuring the geometric quantum discord. *J. Phys. A: Math. Theor.* **45**, 115308 (2012).
28. Yu, C. S., Zhang, J. , Fan, H.: Quantum dissonance is rejected in an overlap measurement scheme. *Phys. Rev. A* **86**, 052317 (2012).
29. Dakic, B., Vedral, V. and Brukner, C.: Necessary and Sufficient Condition for Nonzero Quantum Discord. *Phys. Rev. Lett.* **105**, 190502 (2010).
30. Yu, C. S., *et al.*: Entangling power in deterministic quantum computation with one qubit. *Phys. Rev. A* **87**, 022322 (2013).

